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December 2002

WICAT TR 02-011

WICAT
WIRELESS INTERNET
Center for Advanced Technology

From Power Levels to Power Ratios and Back: A Change of Domain and its Modeling Implications*

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March 15, 2002

Abstract

In certain situations, one wishes to optimize an objective function which depends on the transmit power levels of a certain number of active transceivers. Rather than dealing with the transmit power levels directly, one may wish to choose directly the quantities representing the ratios of each transceiver's power level to the sum of the other transceivers' power levels (plus noise power if applicable). Here we give a closed-form solution for the power vector in terms of these ratios, and discuss which constraints, besides non-negativity, need to be applied to these power ratios to ensure that they correspond to feasible power levels. The interpretation of these results provides some valuable insights applicable to the modeling and analysis of power control problems.

1 Introduction

In certain situations of interest concerning wireless communication networks, one may be interested in optimizing an objective function which depends on the transmit powers of a certain number of active transceivers.

For instance, in certain power control problems, one would like to determine a power level for each active transmitter in such a manner that a suitable measure of network performance, such as the throughput, be optimized. One can set up the pertinent optimization problem in terms of a set of variables which includes one variable representing the transmit power of each active transceiver. However, since, the power levels typically enter the objective function strictly through the ratio of each transceiver's power to the sum of the power of the other active transceivers (plus noise power, if applicable), it may be appropriate for one to replace each variable representing a transceiver's power, for another variable representing the corresponding power ratio.

*Supported in part by the NSF through the grant "Multimodal Collaboration Across Wired and Wireless Networks", and by the NY State OSTAR through the Wireless Internet CAT.

This change of variables may yield some technical advantages, and could provide some valuable insights into the power-control problem, insights which may be missed otherwise. The advantages of this change of domain are discussed further in [4], and it has actually been done in the development of [5], a published paper, as well as in [2], a report of work in progress.

It should be apparent to the reader that the power ratios cannot all be set independently. That is, appropriate constraints (in addition to non-negativity) need to be imposed on the new variables so that their values correspond to feasible (non-negative) power levels. The primary focus of this note is the derivation of a closed-form expression for the power vector in terms of the ratios, which determine explicitly these constraints in a "closed-form" relation.

The interpretation of these results provide some valuable insights into this problem, and have some significant modeling implications.

2 From power ratios to power levels : closed-form solution

As discussed above, in certain situations of interest concerning wireless communication networks, rather than dealing with power levels directly, one may wish to choose directly the quantities representing the ratios of each transceiver's power level to the sum of the other transceivers' power levels (plus noise power if applicable). Here we discuss which constraints, besides non-negativity, need to be applied to these power ratios to ensure that they correspond to feasible power levels, and provide a closed-form relation giving the power vector in terms of the ratios.

Specifically, let α_i be defined as :

$$\alpha_i = \frac{P_i h_i}{\sum_{\substack{j=1 \\ j \neq i}}^{N} P_j h_j + \sigma^2} = \frac{Q_i}{\sum_{\substack{j=1 \\ j \neq i}}^{N} Q_j + \sigma^2} \tag{1}$$

In this expression, P_i is associated with the transmit power of transceiver *i*, h_i corresponds to the path gain from transceiver *i* to the base station, and σ^2 represents the noise power in the base station receiver. $Q_i = P_i h_i$ is then the received power at the base station in the signal transmitted by transceiver *i*. N represents the number of active users.

Each α_i can be called a transceiver's carrier to interference ratio (CIR). The corresponding signal to interference and noise ratio, SINR is defined as the product $G_i \alpha_i$, with G_i the corresponding "processing gain" or ratio of the channel's "chip rate" to the transceiver's transmission rate.

Notice that implicit in the above formulation is the assumption that a single base station is of interest. Considering multiple base stations complicates the notation, without casting any new light on the problem. Hence, a single base station is considered.

The defining equations for the α_i's (see equation (1) above) yield a linear system of equations (see eq. (2) below). The α_j's correspond to feasible power ratios, whenever this system can be solved for physically meaningful Q_j's.

One could attack this issue via elementary algebra. However, a matrix algebra approach, centered on the concepts of eigenvalues and eigenvector, is preferred, because it provides more valuable insights into the structure of the problem. This is not surprising. Eigenvalues and eigenvectors have played a prominent role in the development of power control theory. For instance, see reference [1], which is an influential work.

2.1 Problem formulation

The defining equations for the α_i's (see equation (1) above) yield a system of equations which can be expressed in matrix form as:

$$\begin{pmatrix} 1 & -\alpha_1 & -\alpha_1 & \cdots & -\alpha_1 \\ -\alpha_2 & 1 & -\alpha_2 & \cdots & -\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_N & -\alpha_N & -\alpha_N & \cdots & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} \sigma^2 \tag{2}$$

However, it will prove convenient to divide both sides of each one of the original equations by $1+\alpha_j$ where the index "j" corresponds to the one α_j which appears in the corresponding equation. Thus, for example, the equation corresponding to the second row becomes:

$$-\frac{\alpha_2}{1+\alpha_2}Q_1 + \frac{1}{1+\alpha_2}Q_2 - \frac{\alpha_2}{1+\alpha_2}Q_3 - \cdots - \frac{\alpha_2}{1+\alpha_2}Q_N = \frac{\alpha_2}{1+\alpha_2}\sigma^2 \tag{3}$$

Now, for notational convenience, we define:

$$a_k = \frac{\alpha_k}{1+\alpha_k} \tag{4}$$

It will prove useful to observe the following trivial algebraic identity:

$$a_k + \frac{1}{1+\alpha_k} = \frac{\alpha_k}{1+\alpha_k} + \frac{1}{1+\alpha_k} = 1 \quad \Rightarrow \quad \frac{1}{1+\alpha_k} = 1 - a_k \tag{5}$$

Taking into account (4) and (5), equation (3) can be re-written as

$$-a_2Q_1 + (1-a_2)Q_2 - a_2Q_3 - \cdots - a_2Q_N = a_2\sigma^2$$

After treating all the equations in the system of interest analogously, we can express the system of equations (2) as:

$$\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} - \begin{pmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_N & a_N & a_N & \cdots & a_N \end{pmatrix} \right) \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \sigma^2 \tag{6}$$

Notice that this matrix equation can be expressed as

$$(\mathbf{I} - \mathbf{A})\,\vec{Q} = \vec{a}\sigma^2$$

Above, $\mathbf{I}$ is the $N \times N$ identity matrix, and $\mathbf{A}$ is a strictly positive matrix with each of its columns equal to the vector $\vec{a}$.

2.2 Feasibility Condition

According to non-negative matrix theory, the above system has a non-negative solution whenever the Perron eigenvalue of $\mathbf{A}$ is less than 1. See [3, Section 2.1]. Hence, we need to find the largest eigenvalue of $\mathbf{A}$.

By definition, $\lambda, \vec{v}$ are an eigenvalue/eigenvector pair for matrix $\mathbf{A}$ if $\vec{v} \neq \vec{0}$ and they satisfy:

$$\mathbf{A}\vec{v} = \lambda\vec{v} \tag{7}$$

But because of the special structure of $\mathbf{A}$ it can be easily verified that, for any vector $\vec{x}$,

$$\mathbf{A}\vec{x} = \left(\sum_{j=1}^{N} x_j\right)\vec{a} \tag{8}$$

Comparing eqs. (7) and (8), it becomes apparent that in order for the scalar/ vector pair $\lambda, \vec{v}$ to satisfy eq. (7), it must be that $\lambda = \sum_{j=1}^{N} x_j$ and, either of the following two conditions hold:

i) If $\sum_{j=1}^{N} x_j \neq 0$, $\vec{x}$ must be a multiple of $\vec{a}$ (so that eq. (8) is satisfied).

ii) $\vec{x}$ is a non-zero vector such that $\sum_{j=1}^{N} x_j = 0$ (which would also satisfy eq. (8)).

Notice that, in general, in R^N, one can find $N-1$ linearly independent vectors such that $\sum_{j=1}^{N} x_j = 0$

This development completely specifies the eigenvalues and characterizes the eigenvectors of $\mathbf{A}$.

There are only two distinct eigenvalues : $\lambda_1 = s \doteq \sum_{j=1}^{N} a_j$ and $\lambda_2 = 0$, the latter of which has multiplicity $N-1$. By definition, s is the Perron eigenvalue of $\mathbf{A}$.

The eigenvector corresponding to s can be taken to be precisely $\vec{a}$.

In conclusion, the set of values denoted as $\alpha_j' s$ correspond to feasible power ratios whenever $\sum_{j=1}^{N} a_j < 1$.

2.3 Explicit Solution

One can reach the above conclusion without invoking non-negative matrix theory, by solving explicitly the system of equations of interest: $(\mathbf{I} - \mathbf{A})\vec{Q} = \vec{a}\sigma^2$. This can be done with the information obtained through the preceding development. One has to consider separately two cases: $\sigma > 0$ and $\sigma = 0$.

The case $\sigma > 0$

As discussed above, the right-hand side of the above equation is an eigenvector for the matrix $\mathbf{A}$ corresponding to the eigenvalue $s \doteq \sum_{j=1}^{N} a_j$. This suggests that the relationship between the eigenvalues/eigenvectors pairs of $\mathbf{A}$ and those of the matrix $\mathbf{I} - \mathbf{A}$ be investigated.

In fact, if $\vec{v}$ is and eigenvector for $\mathbf{A}$ corresponding to the eigenvalue λ, then

$$(\mathbf{I} - \mathbf{A})\vec{v} = \vec{v} - \lambda\vec{v} = (1 - \lambda)\vec{v}$$

Thus, $1 - \lambda$ and the same $\vec{v}$ are also an eigenvalue/eigenvector pair for $\mathbf{I} - \mathbf{A}$!

In particular, $\vec{a}$ is also an eigenvector for $\mathbf{I} - \mathbf{A}$ with eigenvalue $1 - s$, that is,

$$(\mathbf{I} - \mathbf{A})\vec{a} = (1 - s)\vec{a}$$

From this it follows that, if $s \neq 1$, $\vec{Q} = (\sigma^2/(1-s))\vec{a}$ (which is positive whenever $\vec{a}$ is, and $s < 1$) is the solution of $(\mathbf{I} - \mathbf{A})\,\vec{Q} = \vec{a}\sigma^2$. That is, each component of the power vector must satisfy

$$\frac{\sigma^2}{1-s} a_k \tag{9}$$

with $s = \sum_{j=1}^{N} a_j$. This expression is well-defined and physically meaningful whenever $s < 1$.

The case $\sigma = 0$

If noise is negligible, which implies $\sigma = 0$, then in order for the system $(\mathbf{I} - \mathbf{A})\,\vec{Q} = \vec{0}$ to have a non-trivial solution, the determinant of the matrix $\mathbf{I} - \mathbf{A}$ must be zero. This can only happen if $1 - s$, which is the only eigenvalue of $\mathbf{I} - \mathbf{A}$ which is different from one (see preceding discussion about the relationship between the eigenvalues of A and those of $\mathbf{I} - \mathbf{A}$), equals zero; that is, if $s = \sum_{j=1}^{N} a_j = 1$.

In this case the system has infinitely many solutions. It can be verified that any power vector $\vec{Q}$ proportional to $\vec{a}$ is a solution to $(\mathbf{I} - \mathbf{A})\,\vec{Q} = \vec{0}$.

In fact, this has already been established. s and $\vec{a}$ have been shown to be and eigenvalue/eigenvector pair for $\mathbf{A}$. That is, $\mathbf{A}\vec{a} = s\vec{a}$. Therefore, when $s = 1$, $(\mathbf{I} - \mathbf{A})\,\vec{a} = \vec{a} - \vec{a} = \vec{0}$, which confirms that $\vec{Q} = \vec{a}$ is indeed a solution of $(\mathbf{I} - \mathbf{A})\,\vec{Q} = \vec{0}$, as is any $\vec{Q} \propto \vec{a}$.

3 Interpretations and Conclusion

Above, the conditions under which each one of a set of positive numbers corresponds to a transceiver's carrier to interference ratio, CIR, have been given. These conditions have been derived by studying the solution of a system of linear equations engendered by the CIR definition (see eq. (1)). In fact, a closed-form expression yielding the solution has been given (see eq. (9)). The interpretation of these results casts some light on the structure of power control problems, and has some implications for the modeling of these problems.

Much of the preceding development is centered on some new variables. These variables may, at first glance, seem devoid of physical significance. However, a more deliberate look at them reveals that they, and the conditions given in terms of them, can be interpreted in physically significant manner.

Recall that the a_k's were introduced in eq. (4) as

$$a_k = \frac{\alpha_k}{1 + \alpha_k}$$

where the α_k's were defined in eq. (1) as the received CSINR's of certain signals. This means that if the original α_k's are indeed physically meaningful, the a_k's can be expressed in terms of the received power vector as follows:

$$a_k = \frac{\alpha_k}{1 + \alpha_k} = \frac{Q_k/I_k}{1 + Q_k/I_k} = \frac{Q_k}{I_k + Q_k} = \frac{Q_k}{Q_C} \tag{10}$$

Above, I_k is the total interfering power, including noise, experienced by user k, i.e, $I_k = \sum_{\substack{j=1 \\ j \neq k}}^{N} Q_j + \sigma^2$, and Q_C is the total received power, including noise. Thus, our a_k's represent the respective signal's fractional "share" of the total power being received (including noise), or the signal-to-channel ratio, SCR, a physically meaningful quantity.

In fact, a_k can be viewed as a rough "measure" of the channel's "quality" as experienced by user k. If $a_k = 1$, user k's signal power is the only one being received (nor even noise interferes with this signal). This represent an ideal situation, in which any non-negligible amount of power received in this signal will result in error-free transmission. At the other extreme, $a_k = 0$ indicates the worst possible situation from the perspective of user k.

Along these lines, the sum $s = \sum_{j=1}^{N} a_j$ is seen to satisfy

$$s = \sum_{j=1}^{N} a_j = \sum_{j=1}^{N} \frac{Q_j}{Q_C} = \frac{\sum_{j=1}^{N} Q_j}{Q_C} \equiv \frac{\sum_{j=1}^{N} Q_j}{\sum_{j=1}^{N} Q_j + \sigma^2} \tag{11}$$

Now, the condition $s < 1$ is discovered to make plenty of sense. If the original α_k's are indeed physically meaningful, as long as $\sigma^2 > 0$, the numerator in the preceding expression, eq. (11), is definitively less than the denominator, for which s must indeed be less than 1. And if $\sigma^2 = 0$, $s = 1$ must hold.

From eq. (11), it follows that $1 - s$, an expression appearing in eq. (9), which gives the power vector in terms of the ratios, represents the noise's fractional "share" of the total received power, σ^2/Q_C. This shows that when the feasibility conditions are satisfied, eq. (9) is an identity. That is:

$$\frac{\sigma^2}{1-s} a_k = \frac{\sigma^2}{(\sigma^2/Q_C)} \frac{Q_k}{Q_C} \equiv Q_k$$

Finally, this analysis has some implications for the modeling of the phenomenon of interest. A critical step in building a mathematical model is to choose an appropriate set of variables. It is well known that some variables help to uncover the underlying structure of the phenomenon of interest and facilitate its analysis, while a different variable choice may hide important interrelations, and complicate the analysis. For example, in the analysis of linear systems, it is often possible to "diagonalize" a matrix representing a linear transformation through a change of coordinates involving the matrix eigenvectors. This new representation can be quite useful in simplifying the analysis.

The development in this note hints that the signal-to-channel ratio (SCR), defined as the ratio of a received signal power to the total received power in the channel (including noise), a quantity with perfectly clear physical significance, may be the "natural" ratio in the analysis of power-control and related phenomena, as opposed to the SINR, which is the ratio "traditionally" favored in the literature.

Both the CIR and the SCR hold the same "information", and the conversion of one to the other is straightforward. However, a candidate SCR vector can be tested for feasibility simply by checking whether the sum of its components is less than 1 (or, if noise is negligible, whether this sum equals one). Likewise, the power vector yielding a desired, feasible SCR vector is directly proportional to the SCR vector, with the constant of proportionality being a simple, physically meaningful function of the sum of the desired SCR's. And if noise is negligible, then the power vector can be taken to be exactly the same as the SCR vector. Hence, the SCR can be a considerably more convenient variable choice than the CIR.

Of course, there is a reason why the CIR has been favored. In the relatively simple AWGN channel, the bit error probability can be shown to be dependent on the signal-to-noise ratio. But the typical wireless channel is considerably more complicated than an AWGN channel. Modeling its bit error probability as determined by the SINR's is a high-level approximation. A similar approximation in terms of the SCR's could be equally justified (or unjustified).

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