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Maximizing the Ratio of a Generalized Sigmoid to its Argument

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Maximizing the Ratio of a Generalized Sigmoid to its Argument*

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Abstract

The ratio f(x)/x, where f is a real-valued, univariate "S-shaped" function, is shown to be quasi-concave, and to always have a unique global maximizer, which can be identified graphically. The analysis is strictly based on geometrical properties derived from the sigmoidal shape. It imposes no specific algebraic functional form on the sigmoid. This optimization is particularly relevant to a transmitter with a limited supply of energy optimally choosing its transmission power for data communication over a wireless medium in the presence of interference. But the conclusions and/or development herein may interest students of many dynamical systems in which sigmoidal functions play a fundamental role.

Key words: optimization, fractional programming, sigmoidal, nonlinear, logistic, convex, quasi-concave

1 Introduction

Sigmoidal functions are particularly useful, having played a fundamental role in the modeling of a wide variety of interesting phenomena in the physical, biological and social sciences. One reason for their ubiquitousness is that the solution to the differential equation $x'(t) = x(t)(k - \alpha x(t))$ has the sigmoidal shape ("logistic growth"). This equation arises naturally in many dynamical systems. For instance, with $\alpha = 1$, $x(t)$ may denote the size, at time t, of certain population, whose instantaneous growth rate is directly proportional to both its current size, and the difference between this size and the environment's "carrying capacity" (maximal sustainable population size), k. Along this line, Ling and He, [4], discuss "logistic growth" and other models in the biological context, with emphasis on tumor growth. Meyer, et al.,[7] argue that sigmoidal functions may be even more useful than traditionally thought, because in many interesting situations, a complex process whose growth behavior may not seem sigmoidal, can be fruitfully modeled via the superposition of various sigmoidal functions in a single model. They provide examples or suggest applications of this approach in many domains, including ecology, psychology, and socio-technological inquiries.

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Likewise, in computing, sigmoidal functions have played the important role of “activation functions” of processing elements in artificial neural networks.

This note focuses on the maximization of the ratio $f(x)/x$, for any real-valued, univariate function f having the specified “sigmoidal” shape. This ratio may admit different interpretations depending on the context. For example, if $x(t)$ is associated with the “logistic growth” of certain process, the ratio $[x(t) - x(0)]/t$, the “average growth rate” at time t, has the form of the ratio being studied here. In a different context, this ratio may represent a measure in bits per Joule of the “energy efficiency” of communication over a wireless medium. This is discussed further below.

The maximization of a ratio of the form $f(x)/x$ is specifically relevant to a situation in which a transmitter with a limited supply of energy chooses optimally its transmission power for data communication over a wireless medium in the presence of interference. The interference may be caused by random noise, or by other transmitters sharing the medium and acting independently. In such scenario, several scientific publications model each transceiver as choosing its transmission power in order to maximize an objective function of the form $f(x)/x$. In this context, the function f is related to f_s, the “frame-success” function, which gives the probability that the transmission of a data “packet” or “frame” is successful. The sigmoidal shape assumed for the functions considered in this work is consistent with that exhibited by “realistic” “frame-success” functions. Goodman and Mandayam, [3], provide an introductory discussion of several works looking at various aspects related to “wireless data”. Rodriguez and Goodman, [9], specifically discuss why a ratio of the form $[f_s(x) - f_s(0)]/x$ makes sense as an objective function for this situation.

Problems involving the optimization of ratios of functions have been intensively studied in the last few decades, and are commonly called “fractional programming”. These problems arise naturally in many contexts, including macroeconomics, finance, inventory control, and numerical analysis, among others. See references [2, 11] for two very recent surveys of this literature. However, the most general formulations studied in this literature involve ratios of concave and convex functions. In a few cases, the definitions of concavity and/or convexity are relaxed to include a somewhat larger class of functions. But, the sigmoidal functions studied here are, by definition, neither concave nor convex (very loosely speaking they are “half and half”), and are, therefore, excluded from the current fractional programming literature.

This note analyzes the “context-free” maximization of the ratio $f(x)/x$ for any function f having the specified “sigmoidal” shape, and characterizes the optimal solution strictly in terms of geometrical properties derived from this “shape”. Specifically, without imposing any particular algebraic functional form on the considered functions, this note shows that, under the assumptions herein, the solution to this maximization problem exists, is unique, and can be graphically described and determined. Additionally, the ratio $f(x)/x$ is shown to be quasi-concave.

Below, the considered class of functions is formally characterized. Then, the solution to the maximization problem of interest is derived. Subsequently, the quasi-concavity of the ratio is established. Finally, some closing comments are given.

2 Specification of the class of functions of interest

In this section, a formal characterization of the class of functions to be considered is given. Subsequently, some statements immediately following from the basic assumptions are discussed. Additionally, some definitions, terminology, and results relevant to the problem of interest are provided below.

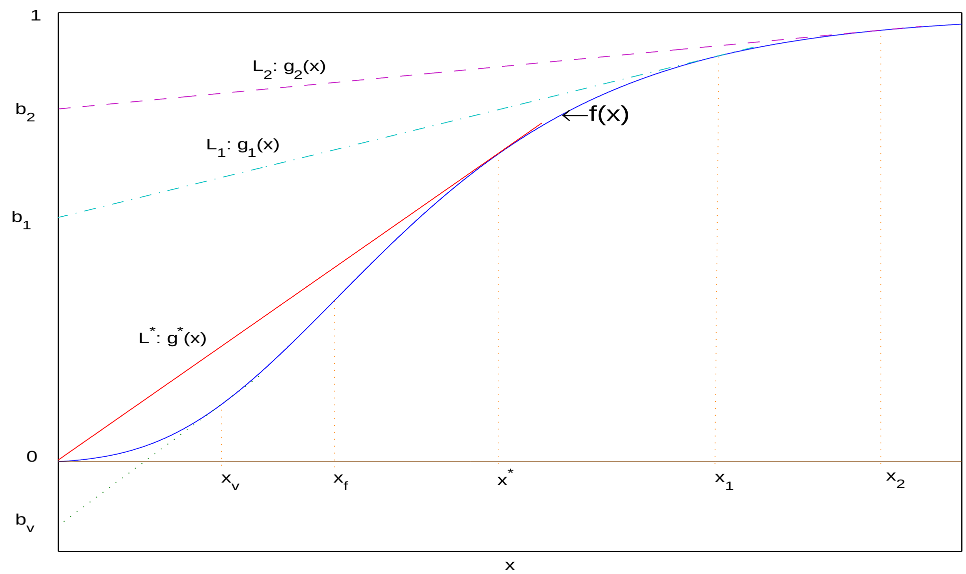

Figure 1: A representative function and some of its tangents

2.1 Basic Assumptions

Figure (1) provides a graphical illustration of a function representative of the class of functions to be considered. Any such function, f, has the following characteristics:

1. Its domain is the non-negative part of the real line; that is, the interval $[0, \infty)$
2. Its range is the interval $[0, B)$, where, for convenience, and without loss of generality, we take $B = 1$.
3. It is increasing.
4. ("Initial convexity") It is strictly convex over the interval $[0, x_f]$, with x_f a positive number.
5. ("Eventual concavity") It is strictly concave over any interval of the form $[x_f, L]$, where L is a positive number greater than x_f
6. It has a continuous derivative.

Notice that no assumptions about the second derivative of the function f are explicitly made.

2.2 Immediately Implied Characteristics

1. Assumptions (1), (2) and (3) imply that $f(0) = 0$.
2. Assumptions (4) ("initial convexity") and (5) ("eventual concavity") imply that the function is continuous for any $x > 0$. (See Theorem 1.3, Chapter III, in reference [1]). And this implication, together with the preceeding one further imply that f is continuous overall.

3. The "initial convexity" assumption (4) and the continuous derivative assumption (6) together imply that $f'(0) < \infty$ (See subsections (2.3.2.1) and (3.2.1)). This ensures that $\lim_{x\to 0} f(x)/x$ is finite, by L'Hopital rule

4. Assumption (6) also implies the continuity of f .

2.3 Relevant Definitions and Results

Much of this as well as other relevant material can be found in reference [1], in particular in chapter III. The presentation here follows that in the mathematical appendix of reference [6]. However, the material of subsection (2.3.2.2) is not found in those references, and is developed in full here.

2.3.1 Concave and convex functions

Consider a function $f : I \to R$, defined on an interval $I \subset \Re$.

Definition: The function f is said to be concave if, $\forall\, x_1, x_2 \in I$ and $\alpha \in (0, 1)$,

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2) \tag{1}$$

The function f is said to be *strictly* concave if the above inequality holds strictly whenever $x_1 \neq x_2$.

Definition: The function f is said to be (strictly) *convex* if the function $-f$ is (strictly) concave.

2.3.2 Properties of continuously differentiable concave and convex functions

2.3.2.1 Tangent line Theorem The continuously differentiable function $f : I \to R$, defined on an interval $I \subset \Re$, is concave if and only if, $\forall\, x_1, x_2 \in I$,

$$f(x_2) \leq f(x_1) + f'(x_1)\cdot(x_2 - x_1) \tag{2}$$

This function is *strictly* concave if and only if the above inequality holds strictly $\forall\,(x_1 \neq x_2) \in I$.

The function f is convex if and only if, $\forall\, x_1, x_2 \in I$,

$$f(x_2) \geq f(x_1) + f'(x_1)\cdot(x_2 - x_1) \tag{3}$$

This function is *strictly* convex if and only if the above inequality holds strictly $\forall\,(x_1 \neq x_2) \in I$.

2.3.2.2 The Monotonicity of y-intercepts *Corollary:* Let $f : I \to R$ denote a continuously differentiable *concave* function, defined on an interval $I \subset \Re$. Let x_0, x_1, x_2 be elements of I such that $x_0 < x_1 < x_2$. Then,

$$f(x_2) + (x_0 - x_2)f'(x_2) \geq f(x_1) + (x_0 - x_1)f'(x_1) \tag{4}$$

If f is *strictly* concave the above inequality holds strictly.

Proof:

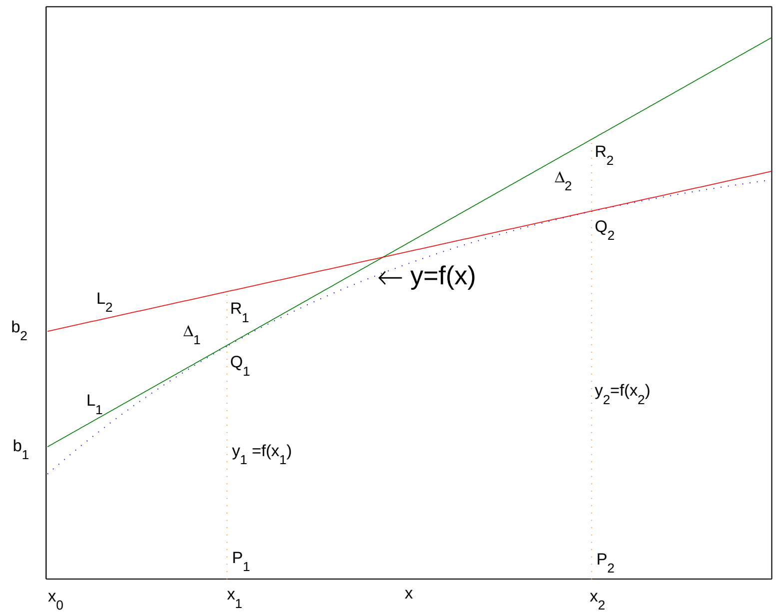

Figure 2: Increasing Y intercepts

See figure (2). In this development, $i \in \{1, 2\}$.
First notice that $g_i(x) = f(x_i) + f'(x_i)(x - x_i)$ denotes the equation of a line tangent at the point (x_i, y_i) $(y_i \doteq f(x_i))$ to the the curve describing the graph of f.
Let $b_i \doteq f(x_i) + (x_0 - x_i)f'(x_i)$.
Thus, b_i is the "height" of tangent line L_i at the abscissa x_0, or its "intercept" with a vertical line drawn at x_0. Hence, inequality (4) can be restated as $b_2 > b_1$. In the special case $x_0 = 0$, b_i become the "y-intercept" or ordinate at the origin of the line L_i.
Let $\Delta_1 \doteq g_2(x_1) - y_1$ and $\Delta_2 \doteq g_1(x_2) - y_2$.
Geometrically, Δ_1 is the length of the segment Q_1R_1, which equals the difference between the "height" of the tangent L_2 and the value of the function f, both measured at the abscissa x_1. Δ_2 has an analogous interpretation.
Observe that the points (x_0, b_1), Q_1 and R_2 are all in the line L_1.
Likewise, (x_0, b_2), R_1 and Q_2 are all in the line L_2.
Therefore:

$$\frac{y_1 - b_1}{x_1 - x_0} = \frac{y_2 + \Delta_2 - b_1}{x_2 - x_0} \Longrightarrow b_1 = \frac{(x_2 - x_0)y_1 - (x_1 - x_0)(y_2 + \Delta_2)}{x_2 - x_1} \tag{5}$$

$$\frac{y_2 - b_2}{x_2 - x_0} = \frac{y_1 + \Delta_1 - b_2}{x_1 - x_0} \Longrightarrow b_2 = \frac{(x_2 - x_0)(y_1 + \Delta_1) - (x_1 - x_0)y_2}{x_2 - x_1} \tag{6}$$

Consequently:

$$b_2 - b_1 = \frac{(x_2 - x_0)\Delta_1 + (x_1 - x_0)\Delta_2}{x_2 - x_1} \tag{7}$$

By construction, $x_0 < x_1 < x_2$.
By inequality (2), both Δ_1 and Δ_2 are non-negative, and both are positive if f is strictly concave.
Therefore, the right hand side of equation (7) is non-negative, and it is positive, if f is strictly concave.
That is, if f is concave, $b_2 \geq b_1$, and $b_2 > b_1$if f is strictly concave.

Q.E.D.

Given the fact that $-f$ is concave whenever f is convex (see section(2.3.1)), the following result is immediate:

Corollary: Let $f : I \rightarrow R$ denote a continuously differentiable *convex* function, defined on an interval $I \subset \Re$. Let $x_0,\ x_1,\ x_2$ be elements of I such that $x_0 < x_1 < x_2$. Then,

$$f(x_2) + (x_0 - x_2)f'(x_2) \leq f(x_1) + (x_0 - x_1)f'(x_1) \tag{8}$$

If f is *strictly* convex the above inequality holds strictly.

3 Maximization

Below, the following optimization problem is solved:

Maximize: $f(x)/x$ subject to $0 \leq x \leq M$

3.1 An interior solution

The development proceeds on the assumption that a "stationary" point exists within the allowable range of x. Subsequently, the possibility that no "stationary" point exists among the feasible values of x is considered.

3.1.1 First-order conditions for a maximum

The first-order necessary conditions are:

$$f(x) - xf'(x) = 0 \tag{9}$$

For the subsequent development, it will prove useful to observe that the equation of a straight line tangent, at the point $(x_1, f(x_1))$, to the curve described by the graph of the function f can be written as

$$g_1(x) = f(x_1) + f'(x_1)(x - x_1) \text{ or } g_1(x) = b(x_1) + f'(x_1)x \tag{10}$$

where $b(x) = f(x) - xf'(x)$ represents the ordinate at the origin (y-intercept) of the straight line tangent at the point $(x, f(x))$ to the curve described by the graph of f (see figure (1)).

One can see immediately that the first-order necessary optimizing condition given by equation (9) can be stated as $b(x) = 0$. This is discussed further in section (3.1.5) below.

3.1.2 Existence of a Solution

A solution to equation (9) always exists. This follows from these facts:

i) $b(x) = f(x) - xf'(x)$ is a continuous function.

ii) For sufficiently large x_L, $b(x_L) > 0$

iii) For any x_v in $(0, x_f]$, $b(x_v) < 0$.

Statement (i) follows directly from the fact that both $f(x)$ and $f'(x)$ have been assumed to be continuous.

Statement (ii) is a direct consequence of the fact that, by assumption, $\lim_{x\to\infty} f(x) = 1$. Hence, in the limit, the tangent line to the graph of f is the line $y = 1$. The y-intercept of this line is, of course, 1. So, $\lim_{x\to\infty} b(x) = 1$, for which $b(x)$ is bound to take on positive values "sooner or later".

Statement (iii) follows from the essential property of tangent lines of continuously differentiable strictly convex functions (see section (2.3.2.1) above). Over the interval $[0, x_f]$, f is assumed to be strictly convex. Taking $x_2 = 0$ and x_1 equal to an arbitrary number in $(0, x_f]$, denoted as x_v, inequality (3) yields $f(0) > f(x_v) + f'(x_v) \cdot (0 - x_v)$ or, equivalently, $b(x_v) = f(x_v) - x_v f'(x_v) < 0$.

Statements (i), (ii), and (iii) above have been shown to be valid. These three facts imply the existence of an x^*satisfying $b(x^*) = 0$, because a continuous function cannot go from a negative to a positive value without taking on the value zero.

Furthermore, notice that the validity of statement (iii) immediately implies that any such x^*must be greater than x_f (that is, any such x^*must be in the interval over which f is concave), since, for any x less than x_f, $b(x) < 0$.

3.1.3 Uniqueness of the solution

In subsection (3.1.2) it was established that any solution to $b(x) = f(x) - xf'(x) = 0$ must be greater than x_f; i.e., it must lie inside the interval where f is strictly concave. The uniqueness of this solution follows directly from the "monotone intercepts" corollary, presented in subsection (2.3.2.2). This results indicates that if x_1 and x_2 are points in an interval of the real line over which the function f is strictly concave, then $x_2 > x_1$implies that $b(x_2) > b(x_1)$. Hence, if $x^*$is such that $b(x^*) = 0$, any $x \neq x^*$must be such that $b(x) \neq 0$

3.1.4 Optimality of the solution

This subsection establishes that the unique solution of equation (9), if feasible, is the global maximizer.

The derivative of the ratio $f(x)/x$ can be expressed as

$$\frac{xf'(x) - f(x)}{x^2} = -\frac{b(x)}{x^2} \tag{11}$$

where, as before, $b(x)$ denotes the "y-intercept" of the straight line which is tangent at the point $(x, f(x))$ to the curve representing the function f. The derivative is well-defined with the possible exception of the boundary value $x = 0$. The case $x = 0$ is discussed in the subsection (3.2). For the purposes of this section, x is assumed to be positive.

The monotone intercepts corollary of subsection (2.3.2.2) specifies that for any $x > x^*$, $b(x) > b(x^*) = 0$. Therefore, the ratio $f(x)/x$ is strictly decreasing for any $x > x^*$. This implies that, for any such x, $f(x)/x < f(x^*)/x^*$.

The same argument leads to the conclusion that the ratio $f(x)/x$ is strictly increasing for any $x_f < x < x^*$. From this, it immediately follows that $f(x)/x < f(x^*)/x^*$ for any x such that $x_f < x < x^*$.

In subsection (3.1.2) it was established that $b(x) < 0$ for any x in $(0, x_f]$. Therefore, the derivative of the ratio $f(x)/x$ is positive for any such x, (see equation (11) above), which means this ratio is increasing over $(0, x_f]$. Hence, for any $x \in (0, x_f)$, $f(x)/x < f(x_f)/x_f < f(x^*)/x^*$

In conclusion, the ratio $f(x)/x$ is less than $f(x^*)/x^*$ for any positive $x \neq x^*$.

3.1.5 Description of the solution: The characteristic tangent

The solution to the first-order necessary optimizing conditions given by equation (9) can be directly identified in the graph of the function f. As has been discussed, only one value, x^*, satisfies equation (9). $(x^*, f(x^*))$ is the only point at which a line tangent to the curve describing the function passes through the origin. Thus, the equation of any such tangent line is $g^*(x) = f'(x^*)x$. (For a graphical illustration, see the tangent line drawn at x^* in figure (1)).

Because of its importance in the determination of the optimal solution, the tangent line above is termed here : the *characteristic tangent* of a given sigmoidal function. Of course, different sigmoids may have the same characteristic tangent.

The value of the objective function at the solution, x^*, can be obtained graphically as the slope of the characteristic tangent, which is $f(x^*)/x^*$. This observation can be useful for conceptual “sensitivity analyses”. The effect on the optimal solution of changing one sigmoid for another (for example via a change in certain parameter) immediately manifests itself, visually, through the new characteristic tangent, and its slope.

3.2 “Boundary” solution

The development so far has ignored the constraint that $x \leq M$ for some M. Below, this issue is addressed. Before that, the possibility that the optimal value be zero is formally discarded.

3.2.1 The non-optimality of x=0

By construction, and the application of L’Hopital rule, $\lim_{x\to 0} f(x)/x = f'(0) < \infty$. In sub-sections (3.1.2) and (3.1.4) it was discussed why the ratio $f(x)/x$ is increasing over the interval $(0, x_f]$. Hence, $x = 0$ is *not* the maximizer.

3.2.2 The global optimality of the smallest of M and x^*

Given the discussion in subsections (3.1.4) and (3.2.1), it is clear that the ratio $f(x)/x$ is increasing over the interval $[0, x^*]$, where x^* is the only value of x satisfying the first-order necessary optimizing conditions given by equation (9). Hence, if the maximum allowable value for x, denoted as M, is less than x^*, $f(M)/M$ is the highest achievable value for the ratio $f(x)/x$. But if x^* is less than M, $x = x^*$ is clearly the optimizing choice. Therefore, under the assumptions stated in section (2.1), the smallest of the numbers M and x^* is the global optimizer for the problem:

Maximize $f(x)/x$, subject to $0 \leq x \leq M$

4 The Quasi-concavity of f(x)/x

In the preceeding development, it has been determined that, for the class of functions under consideration (see section(2.1)), the ratio $f(x)/x$ is "single-peaked"; that is, there is a number x^* such that this ratio is strictly increasing for all $x \in [0, x^*)$ and strictly decreasing for all $x \in (x^*, \infty)$. This implies the quasi-concavity of this ratio. For a general discussion about quasi-concavity and various related concepts and results, see [8].

Below, following the definition of quasi-concavity is given, and the compliance of $f(x)/x$ with this definition is formally established.

4.1 Definition of Quasi-concavity

Definition: The function $h : I \rightarrow R$, defined on an interval $I \subset \Re$, is said to be quasi-concave if its upper contour sets, $\{x \in I \ : \ h(x) \geq t\}$, are convex sets; that is, for any $t \in \Re$, any $\alpha \in [0, 1]$, and any $x_1, x_2 \in I$,

$$h(x_1) \geq t \text{ and } h(x_2) \geq t \text{ implies } h(\alpha x_1 + (1 - \alpha)x_2) \geq t \tag{12}$$

The function h is said to be *strictly* quasi-concave if the implied inequality in (12) holds strictly whenever $x_1 \neq x_2$ and $\alpha \in (0, 1)$.

4.2 Verification of Quasi-concavity

The function $f(x)/x$ is strictly quasi-concave.

For notational convenience, let $h(x) \doteq f(x)/x$ and let $h(x^*) \doteq P^*$.
Let $t \in (0, P^*)$. Notice that verifying (12) is trivial for t outside this interval.
Suppose $0 \leq x_1 < x_2$, $h(x_1) \geq t$ and $h(x_2) \geq t$
Because $h(x)$ is continuous and strictly *increasing* in the interval $[0, x^*)$, there is an x'_t such that $h(x) \geq t$ for all x between x'_t and x^*, and $h(x) < t$ for $x < x'_t$. Likewise, since $h(x)$ is continuous and strictly *decreasing* in the interval (x^*, ∞), there is an x''_t such that $h(x) \geq t$ for all x between $x^*$and x''_t, and $h(x) < t$ for $x > x''_t$.
Then, clearly, any x for which $h(x) \geq t$ must be between x'_t and x''_t, and any x between x'_t and x''_t is such that $h(x) \geq t$. That is, $x'_t \leq x \leq x''_t \Leftrightarrow h(x) \geq t$.
Therefore, $h(x_1) \geq t$ and $h(x_2) \geq t$ implies $x'_t \leq x_1 < x_2 \leq x''_t$
And for $\alpha \in (0, 1)$, $x_1 < \alpha x_1 + (1 - \alpha)x_2 < x_2$. This implies $x'_t < \alpha x_1 + (1 - \alpha)x_2 < x''_t$, which further implies $h(\alpha x_1 + (1 - \alpha)x_2) \geq t$

Q.E.D.

5 Concluding remarks

The “context-free” maximization of the ratio $f(x)/x$ for any function f having a “sigmoidal” shape has been studied, and its optimal solution been characterized without assuming any particular algebraic functional form on the considered functions. “Sigmoidness” has been captured in a strictly geometric manner, by assuming that the considered functions “start out” convex at the origin, and “smoothly” transition to concave as they approach a horizontal asymptote. No prior use of this geometric construction has yet been found in the scientific literature. Then, on the basis of geometrical properties derived from this “shape”, this note shows that, under the assumptions herein, the solution to the maximization problem of interest exists, is unique, and can be graphically described and determined. Along the way, the ratio $f(x)/x$ has been shown to be quasi-concave.

Central to the development herein is the observation that the “y-intercepts” of concave and convex functions are monotonic. Fully developed here, this result may be useful beyond the particular aims of this note.

The proof that the studied ratio is quasi-concave, which is by no means obvious given the arbitrary sigmoidal shape of the function in the numerator, can be beneficial in situations in which this maximization is embedded into a larger problem. In this case, the quasi-concavity of this ratio may allow the use of theorems and results which could not be invoked otherwise.

It has been further observed that this problem’s solution can be obtained graphically, in a single step, simply by drawing a tangent line from the graph of the considered function to the origin. Given today’s computational resources, graphical solution procedures have lost much of their appeal. However, this observation could still be valuable as a conceptual tool to understand the meaning of the solution, as well as a "sensitivity analysis" tool, to visualize how a change in the considered function can impact the optimal solution.

The maximization of a ratio of the form $f(x)/x$ is particularly relevant to a situation in which a transmitter with a limited supply of energy wishes to optimally choose its transmission power for data communication over a wireless medium in the presence of interference, which may be caused by random noise, or by other simultaneous transmitters. But sigmoidal functions have proven to be particularly useful in the modeling of a wide variety of interesting phenomena in the physical, biological and social sciences. This suggests that the results in this note, or at least its development, could be of interest across disciplines to a variegated group of researchers.

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